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PARTICULAR SOLUTIONS $CP(N-1)$ MODEL

(bachelor's work)

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Abstract

The research aims at considering the specific cases of $CP(N-1)$ model in classic field theory and the search of exact solutions. For that, the equations of the motion supplement with restrictions, which simplify finding concrete solutions. The model can rewrite through a convenient matrix form, which promotes to investigate various cases. Some cases analyzed in detail. The significant part of the work dedicates the research of three particular cases in the $1+1$ dimensions. The first case represents the time-harmonic oscillations, in which constants of integration corresponding Hamiltonian density obtained and the link with the Ermakov equation found. The second case describes some exact solutions in $CP(1)$. The third case demonstrates the connection between a specific class of gauge fields with Hopf equation in the $1+1$ dimensions, provides arguments about the possible existence of corresponding topologically nontrivial solutions.

Contents

Designations

- $a_\mu b_\mu := a_0 b_0 - a_i b_i$
- Suppose summation over repeated indices $a_j b_j := \sum_{j=1}^{N} a_j b_j$. We will write $\sum_{j=1}^{N} a_j b_j$, when it's necessary
- $D_\mu = \partial_\mu - i A_\mu,\ \Box = \partial_\mu \partial_\mu,\ \triangle = \partial_i \partial_i$
- $j \in \overline{1, N} \Leftrightarrow j \in \{1, 2, ..., N\}$

Introduction

Theoretical physicists develop mathematical models to describe physical phenomena. The purpose of any physicist is to create a coherent theory explaining experimental data. Therefore, connections between theories are meaningful and valuable. It explains the importance of the investigation nonlinear sigma model, which intertwines with numerous modern branches of theoretical physics. Because of the complexity of the theory and its nonlinearity, people analyze various cases and particular solutions.

Nonlinear sigma model $CP(N-1)$ is a complex scalar field theory with interaction, where N fields form complex projective space. The model is well-researched in two dimensions because of simplicity and applicability.

The main aspects of the $CP(N-1)$ model for quantum chromodynamics (QCD) investigated by Novikov, Shifman, Vainshtein, and Zakharov [1]. They point out the parallel between two-dimensional sigma models and four-dimensional Yang-Mills theory, applying in QCD. $CP(N-1)$ model serves as a toy model for QCD due to existing features like asymptotic freedom, confinement, the generation of the mass gap, and chiral symmetry breaking. Moreover, $CP(N-1)$ can use for examining the validity of low energy theorems in QCD.

Solutions of $CP(N-1)$ are essential for the nonabelian string theory. According to Tong's study [2], $CP(N-1)$ sigma model describes the low-energy dynamics of the string worldsheet. Furthermore, Shifman and Yung showed correspondence between BPS kinks in $CP(1)$ and confined monopoles [3].

$CP(N-1)$ model appears in the quantum field theory (QFT). Gorsky, Pikalov, Vainstein explored the model in two dimensions and showed homogeneous solutions are not the ground state [4]. They obtained new solitonic solutions, which turned out to have less energy than in the homogeneous case. There are a lot of investigations $CP(N-1)$ in QFT in two dimensions on the plane, sphere, cylinder, disc, and annulus [5].

The set of interesting classic solutions Misumi, Nitta, and Sakai find on a cylinder with twisted boundary conditions [6, 7]. These solutions have finite and fractional action and constitute instantons and anti-instantons with a fractional topological number. Some of their works dedicated to the modeling of interaction that particles.

The complex projective plane is well-studied in such branches of math as algebraic geometry and topology. That's why the classic field theory of the $CP(N-1)$ model attracts mathematics attention. Din and Zakrzewski described general solutions on the Riemann sphere with finite action in terms of N rational functions [8].

$CP(N-1)$ model has a link with supersymmetry and numerous applications, such as antiferromagnetic spin chains and quantum Hall effect.

However, there is no comprehensive description of all set solutions. New particular solutions of the $CP(N-1)$ model directly reflect on results in many branches of physics and promote their development. Therefore, the study of new classic and quantum solutions of the $CP(N-1)$ model and their classification is significant for modern theoretical physics.

The **object of research** represents a nonlinear $CP(N-1)$ model in classic field theory.

The **subject of research** is particular solutions of nonlinear $CP(N-1)$ model in classic field theory.

The **purpose of the research** represents the search particular exact solutions of nonlinear $CP(N-1)$ model in classic field theory.

Research problems:

1. Describe the model most conveniently;

2. Define the uncomplicated cases and solve the corresponding system of equations.

The **method of research** is a theoretical analysis of the system of nonlinear differential equations.

The **theoretical significance of the research** represents in search of features of the model which can help to explore more complex cases and quantum cases.

The work divides into three parts. The first part describes the general aspects of the theory. Furthermore, we represent the matrix formulation of the model. The second part focuses on the particular central cases and defines the behavior of solutions. In the third part, we consider three cases in $1+1$ dimensions, in which we receive exact results.

The first case represents the time-harmonic oscillations, in which we obtained constants of integration corresponding Hamiltonian density and discovered the link with the Ermakov equation. The second case describes some exact solutions in the $CP(1)$ model. The third case reveals the connection of a specific class of gauge fields with Hopf equation in the $1+1$ dimension and provides reasons for the possible existence of topologically nontrivial solutions. The links with well-known differential equations give a new look on the model, and the arguments for existence topologically nontrivial solutions can point out the direction for a search of new particular solutions. Furthermore, the solutions of the second and the third cases are valuable for understanding the nature of confined monopoles, which connects with the $CP(1)$ model.

1 General case

Nonlinear $CP(N-1)$ can derive from $O(N)$ sigma model through adding real auxiliary fields A_μ and changing derivatives ∂_μ on the covariant $D_\mu = \partial_\mu - iA_\mu$. That changes provide local gauge invariance.

In that part, we consider primary characteristics of nonlinear $CP(N-1)$ model in classic field theory, describe convenient formulation via square matrices, select the field corresponding for gauge transformations, and obtain an explicit expression for field strength tensor $F_{\mu\nu}$.

The model set uniquely in the classic field theory by the following parameters

- N - number of complex fields;
- $\mathcal{M}$ - the space on which fields define;
- the choice of gauge;
- the choice of boundary conditions.

In this paper we consider

- N or 2 complex fields;
- $\mathcal{M} = \mathbb{R} \times \mathbb{R}^m$ or $(\mathbb{R} \times \mathbb{R})$;
- the Lorentz gauge $\partial_\mu A_\mu = 0$;
- we don't choose definite boundary conditions and only analyze general case

1.1 Basic characteristic

Lagrangian density

Lagrangian of the theory

$$\mathcal{L} = \overline{D_\mu n_j} D_\mu n_j - \Lambda(\bar{n}_j n_j - r) \tag{1.1}$$

$$\mathcal{L} = \partial_\mu \bar{n}_j \partial_\mu n_j + iA_\mu \left(\bar{n}_j \partial_\mu n_j - n_j \partial_\mu \bar{n}_j\right) + rA_\mu A_\mu - \Lambda(\bar{n}_j n_j - r) \tag{1.2}$$

where define $r > 0$. The last term is the Lagrangian multiplier and describes restrictions on fields. In this paper, we consider $\varLambda$ some constant. However, there are hints that this simplification limits us[1]. The Lagrangian (1.1) preserves the local gauge invariance under transformations

$$\begin{aligned} \widetilde{n}_j &= n_j e^{i\alpha(t,x)} \\ \tilde{A}_\mu &= A_\mu + \partial_\mu \alpha \end{aligned} \tag{1.3}$$

[1]If we suppose that $\varLambda$ is not constant in the system of equations in Table case 2 for $CP(1)$ with additional restriction might have nontrivial solutions

If we vary by Λ, we get relations

$$\bar{n}_j n_j = r \tag{1.4}$$
$$\bar{n}_j \partial_\mu n_j = -n_j \partial_\mu \bar{n}_j \tag{1.5}$$

If we vary by A_μ, we get that

$$A_\mu = -\frac{i}{2r}\left(\bar{n}_j \partial_\mu n_j - n_j \partial_\mu \bar{n}_j\right) \tag{1.6}$$

Hamiltonian density
The canonical momentum π_j

$$\pi_j = \frac{\partial \mathcal{L}}{\partial \dot{n}_j} = \dot{\bar{n}}_j + iA_0 \bar{n}_j \tag{1.7}$$

In the Lorentz gauge

$$\partial_\mu A_\mu = 0 \tag{1.8}$$

the Hamiltonian density $\mathcal{H}$ looks

$$\begin{aligned}\mathcal{H} &= \pi_j \dot{n}_j + \bar{\pi}_j \dot{\bar{n}}_j - \mathcal{L} \\ &= \dot{n}_j \dot{\bar{n}}_j + \partial_i \bar{n}_j \partial_i n_j - r\left(A_0^2 + A_i A_i\right)\end{aligned} \tag{1.9}$$

Here we substitute (1.6). The gauge field A_μ reduces the density of Hamiltonian.

Equations of motion

$$-\Box n_j + 2iA_\mu \partial_\mu n_j + i n_j \partial_\mu A_\mu - \Lambda n_j = 0,\ j \in \overline{1, N} \tag{1.10}$$

In the Lorentz gauge (1.8), the equations of motion are

$$-\Box n_j + 2iA_\mu \partial_\mu n_j - \Lambda n_j = 0 \tag{1.11}$$

1.2 Matrix form of the nonlinear $CP(N-1)$ model

Complex diagonal matrices allow rewriting the task more compactly, without indexes. The comparison of the two representations is in Table 1. (E - unit matrix)

Table 1: Two representations of the model

N complex fields $n_j = n_j(t,x) \vert\ \bar{n}_j n_j = r$	Diagonal matrix $M \in GL(N,\mathbb{C}) \vert\ Tr\bar{M}M = r$
$\{n_j(t,x)\}_{j\in\overline{1,N}}$	$M = \begin{pmatrix} n_1(t,x) & 0 & \cdots & 0 \\ 0 & n_2(t,x) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & n_N(t,x) \end{pmatrix}$
Density of the Lagrangian	
$\mathcal{L} = \overline{D_\mu n_j} D_\mu n_j + \Lambda(\bar{n}_j n_j - r)$	$\mathcal{L} = Tr\left(\overline{D_\mu M} D_\mu M + \Lambda(\bar{M}M - \frac{r}{N}E)\right)$
$A_\mu = -\frac{i}{2r}\left(\bar{n}_j \partial_\mu n_j - n_j \partial_\mu \bar{n}_j\right)$	$A_\mu = -\frac{i}{2r} Tr\left(\bar{M}\partial_\mu M - M\partial_\mu \bar{M}\right)$
Density of the Hamiltonian	
$\mathcal{H} = \dot{n}_j \dot{\bar{n}}_j + \partial_i \bar{n}_j \partial_i n_j - r\left(A_0^2 + A_i A_i\right)$	$\mathcal{H} = Tr\left(\dot{\bar{M}}\dot{M} + \partial_i \bar{M}\partial_i M\right) - r\left(A_0^2 + A_i A_i\right)$
Equations of motion	
$-\Box\, n_j + 2iA_\mu \partial_\mu n_j - \Lambda n_j = 0,\ j \in \overline{1,N}$ $n_j = \rho_j e^{i\varphi_j}$	$-\Box\, M + 2iA_\mu \partial_\mu M - \Lambda M = 0$ $M = R\, e^{i\Phi}$

$$M = R\, e^{i\Phi} \tag{1.12}$$

$$R = \begin{pmatrix} \rho_1(t,x) & 0 & \cdots & 0 \\ 0 & \rho_2(t,x) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \rho_N(t,x) \end{pmatrix},\ \Phi = \begin{pmatrix} \varphi_1(t,x) & 0 & \cdots & 0 \\ 0 & \varphi_2(t,x) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \varphi_N(t,x) \end{pmatrix}$$

Notice, that $e^{i\Phi}$ we can consider as a point on N-dimensional tor T^n. With consideration matrix exponential form (1.12), $A_\mu$rewrites

$$A_\mu = \frac{1}{r} Tr\, R^2 \partial_\mu \Phi = \frac{Tr\, R^2 \partial_\mu \Phi}{Tr\, R^2} \tag{1.13}$$

The equations of motion (1.11) rewrite through the system, corresponding imaginary, and real parts

$$\begin{cases} -\Box\, R + \left(\partial_\mu \Phi \partial_\mu \Phi - 2A_\mu \partial_\mu \Phi - \Lambda E\right) R = 0 \\ R\,\Box\,\Phi + 2\left(\partial_\mu \Phi - A_\mu E\right)\partial_\mu R = 0 \end{cases} \tag{1.14}$$

E is the unit matrix. The significant part of this paper dedicates to the study of the nonlinear system of differential equations (1.14) and its particular cases.

1.3 Gauge transformations

Let's consider gauge transformations in two representations in the Table 2

Table 2: Gauge transformations in two representations

N complex fields $n_j = n_j(t,x)$	Diagonal matrix $M \in GL(N,\mathbb{C})$
$\widetilde{n}_j = n_j e^{i\alpha(t,x)}$	$\tilde{M} = M e^{i\alpha(t,x)E}$
$\tilde{A}_\mu \overset{(1.6)}{=} A_\mu + \partial_\mu \alpha$	$\tilde{A}_\mu \overset{(1.13)}{=} A_\mu + \partial_\mu \alpha$

E - unit matrix. This prompts us to select the part of the matrix that responsible for gauge transformations

$$\Phi = \begin{pmatrix} \varphi_1(t,x) & 0 & \cdots & 0 \\ 0 & \varphi_2(t,x) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \varphi_N(t,x) \end{pmatrix} = \tag{1.15}$$

$$= \begin{pmatrix} \frac{Tr\Phi}{N} + \underbrace{\varphi_1(t,x) - \frac{Tr\Phi}{N}}_{\psi_1(t,x)} & 0 & \cdots & 0 \\ 0 & \frac{Tr\Phi}{N} + \underbrace{\varphi_2(t,x) - \frac{Tr\Phi}{N}}_{\psi_2(t,x)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{Tr\Phi}{N} + \underbrace{\varphi_N(t,x) - \frac{Tr\Phi}{N}}_{\psi_N(t,x)} \end{pmatrix} \tag{1.16}$$

In the designation

$$\alpha(t,x) = \frac{Tr\Phi}{N}, \quad \Psi = diag(\psi_1, ..., \psi_N) \tag{1.17}$$

the matrix of phases has the decomposition

$$\Phi = \alpha(t,x)E + \Psi, \; Tr\,\Psi = 0 \tag{1.18}$$

It means that we always can write

$$A_\mu = \partial_\mu \alpha + \frac{1}{r} Tr\, R^2 \partial_\mu \Psi, \quad \begin{matrix} Tr\, R^2 = r \\ Tr\,\Psi = 0 \end{matrix} \tag{1.19}$$

As well as $Tr\,\Psi = 0$ we can use the basis for diagonal traceless matrices

$$\Psi = \sum_{i=1}^{N-1} \phi_i h_i, \; h_i = E_{ii} - E_{i+1,i+1} \tag{1.20}$$

$E_{ii} = diag(0, ..., 0_{i-1}, 1_i, 0_{i+1}, ..., 0)$. From here

$$A_\mu = \partial_\mu \alpha + \frac{1}{r} \sum_{i=1}^{N-1} (\rho_i^2 - \rho_{i+1}^2) \partial_\mu \phi_i \tag{1.21}$$

We can write an explicit expression for $F_{\mu\nu}$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \frac{2}{r} Tr\, R(\partial_\mu R\, \partial_\nu \Psi - \partial_\nu R\, \partial_\mu \Psi) \tag{1.22}$$

$$Tr\, R^2 = r, \quad Tr\, R \partial_\mu R = 0, \quad Tr\, \Psi = 0, \quad Tr\, \partial_\mu \Psi = 0 \tag{1.23}$$

Notice that

1) $F_{\mu\nu}$ doesn't contain gauge term and is gauge invariant;

2) $F_{\mu\nu} = 0$ in cases $R = const$, $\Psi = const$, $R = \Psi$. The non-trivial $F_{\mu\nu}$ can be only when amplitude and phases of fields change.

There is an interesting observation that we can write the formula (1.22) through the two traceless matrices. For that, we represent the matrix R in a new way

$$R^2 = \frac{r}{N} E + P, \; Tr\, P = 0 \tag{1.24}$$

If we use the basis for traceless matrices $P = \sum_{i=1}^{N-1} p_i h_i$, we get restrictions on p_i

$$| p_1 | \leq \frac{r}{N}, \; | p_{N-1} | \leq \frac{r}{N}, \; | p_i - p_{i+1} | \leq \frac{r}{N}, \; i \in \overline{1, N-2} \tag{1.25}$$

With that restriction, we can rewrite (1.22) through the two traceless matrices

$$\begin{gathered} F_{\mu\nu} = \tfrac{1}{r} Tr\, (\partial_\mu P\, \partial_\nu \Psi - \partial_\nu P\, \partial_\mu \Psi) \\ Tr\, P = 0, \; Tr\, \Psi = 0 \end{gathered} \tag{1.26}$$

And in that case, the gauge field A_μ has the form

$$A_\mu = \partial_\mu \alpha + \frac{1}{r} Tr\, P \partial_\mu \Psi, \quad \begin{gathered} Tr\, P = 0 \\ Tr\, \Psi = 0 \end{gathered} \tag{1.27}$$

We can consider $F_{\mu\nu}$ as a skew-symmetric form from the two traceless matrices P and Ψ

$$F_{\mu\nu} = \langle P, \Psi \rangle_{\mu\nu} \tag{1.28}$$

Obtained formulas demonstrate the convenience of the matrix representation. Further research on the role of the basis for traceless matrices for the nonlinear system of equations and the $CP(N-1)$ model is necessary.

Now we conveniently rewrite the model. The system of equations (1.14) has a complicated view, so it is a point to investigate extreme cases.

2 The central specific cases

Consider the matrix form of the model in $1+m$ dimensions with N complex fields

$$M = R\, e^{i\alpha E + i\Psi}$$
$$Tr\, R^2 = r,\; Tr\, \Psi = 0 \tag{2.1}$$

The representation (2.1) has 3 central parameters R, α and Ψ. As well as equations of motion quite complicated, we can consider different cases, which all lead in the Table 3, to catch on the main dependencies. The cases differ in what we are going to deem as a function or constant. (C - constant, F - function)

Table 3: The central specific cases

	R	α	Ψ	A_μ	$F_{\mu\nu}$	Equation of motion
1)	C	C	C	0	0	Fulfilled
2)	C	C	F	$\frac{1}{r}Tr\, R^2 \partial_\mu \Psi$	0	$\begin{cases} \partial_\mu \Psi \partial_\mu \Psi - 2A_\mu \partial_\mu \Psi = \Lambda E \\ \Box \Psi = 0 \end{cases}$
3)	C	F	C	$\partial_\mu \alpha$	0	$\begin{cases} \partial_\mu \alpha \partial_\mu \alpha + \Lambda = 0 \\ \Box \alpha = 0 \end{cases}$
4)	C	F	F	(1.19)	0	$\begin{cases} \partial_\mu \Phi \partial_\mu \Phi - 2A_\mu \partial_\mu \Phi = \Lambda E \\ \Box \Phi = 0 \end{cases}$
5)	F	C	C	0	0	$(\Box + \Lambda)R = 0$
6)	F	C	F	$\frac{1}{r}Tr\, R^2 \partial_\mu \Psi$	(1.22)	$\begin{cases} -\Box\, R + (\partial_\mu \Psi \partial_\mu \Psi - 2A_\mu \partial_\mu \Psi - \Lambda E)\, R = 0 \\ R\, \Box\, \Psi + 2\,(\partial_\mu \Psi - A_\mu E)\, \partial_\mu R = 0 \end{cases}$
7)	F	F	C	$\partial_\mu \alpha$	0	$\begin{cases} \Box R + (\partial_\mu \alpha \partial_\mu \alpha + \Lambda) R = 0 \\ \Box \alpha = 0 \end{cases}$
8)	F	F	F	(1.19)	(1.22)	$\begin{cases} -\Box\, R + (\partial_\mu \Phi \partial_\mu \Phi - 2A_\mu \partial_\mu \Phi - \Lambda E)\, R = 0 \\ R\, \Box\, \Phi + 2\,(\partial_\mu \Phi - A_\mu E)\, \partial_\mu R = 0 \end{cases}$

Let's look at some cases.

2) Consider substitutions

I. $\Phi = \Phi(\omega_\mu x_\mu)$, $\omega_\mu = const\,|\,\omega_\mu \omega_\mu = 0$. Then the upper equation of the system leads $\Lambda = 0 \Rightarrow$ contradiction. It means that the case implies different frequencies of the fields.

II. $\Psi = \Omega_\mu x_\mu$, $Tr\, \Psi = 0$, $\Omega_\mu = diag(\omega^1_\mu, ..., \omega^N_\mu) = const\,|\,\Omega_\mu \Omega_\mu = 0$. It is the simple case of plane waves with different frequencies. We have $A_\mu = const$, and the upper equation is the only restriction on constants $A_\mu \Omega_\mu = -\frac{\Lambda}{2} E$

III. Consider the case in $\mathbb{R}^m$ spacial dimensions[2] with $m > 1$.

[2] When $m = 1$, the reasoning is not right. The explanation is in Connection of the gauge field with Hopf equation

$$\Psi = \begin{pmatrix} \psi(\omega^1_\mu x_\mu) & 0 & & \cdots & 0 \\ 0 & \psi(\omega^2_\mu x_\mu) - \psi(\omega^1_\mu x_\mu) & & & \vdots \\ & & \ddots & & \\ \vdots & & & \psi(\omega^{N-1}_\mu x_\mu) - \psi(\omega^{N-2}_\mu x_\mu) & 0 \\ 0 & \cdots & & 0 & -\psi(\omega^{N-1}_\mu x_\mu) \end{pmatrix}$$

Suggest $\omega^1_\mu \omega^1_\mu = 0$, $(\omega^j_\mu - \omega^{j-1}_\mu)(\omega^j_\mu - \omega^{j-1}_\mu) = 0$, $\omega^{N-1}_\mu \omega^{N-1}_\mu = 0$, then $\partial_\mu \Psi \partial_\mu \Psi = 0$ and

$$-2A_\mu \partial_\mu \Phi = \Lambda E \tag{2.2}$$

If we multiply on R^2, take the trace and remind view of A_μ, we get

$$A_\mu A_\mu = -\frac{\Lambda}{2} \tag{2.3}$$

This case takes place when the dimension of the space higher than the number of fields N. It provides assumptions fulfilled.

3) Particular solutions of the system can be plane waves

$$\alpha = \omega_\mu x_\mu,\ \omega_\mu \omega_\mu = -\Lambda,\ \omega_\mu = const \tag{2.4}$$

5) Each field ρ_j should satisfy the Klein Gordon Fock equation. The parameter $\Lambda = m^2$ corresponds to the mass of the field. In that case, we have N fields with the same mass $\sqrt{\Lambda}$.

7) That case can join 3) and 5). If $\alpha = \omega_\mu x_\mu$, $\omega_\mu = const$, then we have N fields with the same mass $\sqrt{\Lambda + \omega_\mu \omega_\mu}$. So through the gauge field α, we can change the mass of all fields. In particular, if $\omega_\mu \omega_\mu = -\Lambda$, then we have $\Box R = 0$, and solutions for fields are real massless plane waves.

Those cases are useful because they allow us to comprehend what we should get in different extreme cases. Besides, we see the intricacy of the theory lies in the equations for the phases of fields Ψ.

The possible continuation of the work can be an analysis of all specific cases for the $CP(1)$ model in $1+1$ dimensions. The idea is to construct general solutions by joining particular cases.

3 Specific cases in 1+1 dimension

3.1 $CP(N-1)$ and time-harmonic oscillations

The purpose of this part is to investigate time-harmonic oscillations. For the 1+1 space-time dimension, accurate solutions obtained. It has found interesting constants of integration similar Hamiltonian density and angular momentum and has found a link with nonlinear Ermakov equation.

Transition to real fields

In that part, we change $\Lambda \to -\Lambda$. At first, we make a transition to the real and imaginary parts of fields.

Denote $n_j(t,x) = \boldsymbol{u_j}(t,x) + i\boldsymbol{v_j}(t,x)$. The gauge field expresses

$$A_\mu \overset{(1.6)}{=} \frac{1}{r}(u_j\partial_\mu v_j - v_j\partial_\mu u_j) \tag{3.1}$$

The Hamiltonian density is

$$\begin{aligned}\mathcal{H} &= \dot{n}_j\dot{\bar{n}}_j + \partial_i\bar{n}_j\partial_i n_j - rA_0^2 - rA_i^2 \\ &= \dot{u}_j^2 + \dot{v}_j^2 + (u_j')^2 + (v_j')^2 - rA_0^2 - rA_i^2\end{aligned} \tag{3.2}$$

The equations of motion are

$$\begin{cases} -\Box\, u_j - 2A_\mu\partial_\mu v_j + \Lambda u_j = 0 \\ -\Box\, v_j + 2A_\mu\partial_\mu u_j + \Lambda v_j = 0 \end{cases} \tag{3.3}$$

3.1.1 Special substitution

Consider the particular case of time-harmonic oscillations in $1+1$ dimensions

$$n_j(t,x) = (u_j(x) + iv_j(x))e^{i\omega t},\ \boldsymbol{\omega} = const \in \mathbb{R} \tag{3.4}$$

Such substitution allows us to obtain an exact solution for fields n_j.

For (3.4) we get that $A_0 \overset{(1.6)}{=} \omega \ \Rightarrow\ A_0 = const \in \mathbb{R} \Rightarrow$ the equivalent case is

$$n_j(t,x) = (u_j(x) + iv_j(x))e^{iA_0 t},\ A_0 = const \in \mathbb{R} \tag{3.5}$$

From the choice of gauge (1.8) follows $A_1' = 0$ and

$$A_1 = const \in \mathbb{R} \tag{3.6}$$

Substitute (3.5) in equations of motion (1.11) and get the system

$$u_j'' + 2A_1 v_j' + (\Lambda - A_0^2)u_j = 0 \tag{3.7}$$

$$v_j'' - 2A_1 u_j' + (\Lambda - A_0^2)v_j = 0 \tag{3.8}$$

If we multiply (3.7) on u^j, (3.8) on v^j, sum up the obtained equations by j and use (3.1), we get

$$\sum_{j=1}^{N} \left(u_j u_j'' + v_j v_j''\right) + 2rA_1^2 + (\Lambda - A_0^2)r = 0 \tag{3.9}$$

Remind that

$$(u_j)^2 + (v_j)^2 = r \quad \Rightarrow \quad \begin{aligned} & u_j u_j' + v_j v_j' = 0 \\ & u_j u_j'' + v_j v_j'' + u_j' u_j' + v_j' v_j' = 0 \end{aligned} \tag{3.10}$$

In (3.10) differentiation by time or coordinate. So, using (3.10) we get a value relation

$$\sum_{j=1}^{N} (u_j')^2 + (v_j')^2 = 2rA_1^2 + (\Lambda - A_0^2)r \tag{3.11}$$

Let's multiply (3.7) on u_j', (3.8) on v_j', sum obtained equations and get

$$\left((u_j')^2 + (v_j')^2 + (\Lambda - A_0^2)(u_j^2 + v_j^2)\right)' = 0 \tag{3.12}$$

Define

$$E_j(x) = \frac{1}{2}\left((u_j')^2 + (v_j')^2 + (\Lambda - A_0^2)(u_j^2 + v_j^2)\right), \; E_0 = \sum_{j=1}^{N} E_j \tag{3.13}$$

We see that the view of E_j reminds energy for the harmonic oscillator. The (3.12) leads to

$$E_j = const \tag{3.14}$$

$$E_j = \frac{1}{2}\underbrace{\left((u_j')^2 + (\Lambda - A_0^2)u_j^2\right)}_{U^j(x)} + \frac{1}{2}\underbrace{\left((v_j')^2 + (\Lambda - A_0^2)v_j^2\right)}_{V^j(x)} = const \in \mathbb{R}\, \forall j \in \overline{1, N} \tag{3.15}$$

$$U_j(x) + V_j(x) = 2E_j \tag{3.16}$$

$$E_0 = \sum_{j=1}^{N} E_j = \frac{1}{2}\sum_{j=1}^{N} (u_j')^2 + (v_j')^2 + (\Lambda - A_0^2)(u_j^2 + v_j^2) \overset{(3.11)}{=} r(A_1^2 - A_0^2 + \Lambda)$$

$$E_0 = r(\Lambda - A_\mu A_\mu) = r\Omega^2 \tag{3.17}$$

where $\Omega^2 = \Lambda - A_\mu A_\mu$. Suppose $\Lambda - A_0^2 > 0$. Then $E_j > 0$, $\forall j \in \overline{1, N}$

$U_j = const$

There is no mixing of fields between the imaginary and real parts.

$$U_j(x) = const \Rightarrow V_j(x) = 2E_j - U_j = const$$

Then

$$(u_j')^2 + (\Lambda - A_0^2)u_j^2 = U_j \tag{3.18}$$

In that case, the equations can integrate, and we get an explicit view of functions $u_j(x)$ and $v_j(x)$

$$\begin{cases} u_j(x) = \sqrt{\frac{U_j}{\Lambda - A_0^2}}\, sin(x\sqrt{\Lambda - A_0^2} + \xi_j) \\ v_j(x) = \sqrt{\frac{V_j}{\Lambda - A_0^2}}\, sin(x\sqrt{\Lambda - A_0^2} + \zeta_j) \end{cases} \tag{3.19}$$

From the substitution obtained functions (3.19) in (3.1) we get that $A_1 = 0$. It means that $\Omega^2 = \Lambda - A_\mu A_\mu = \Lambda - A_0^2$ and equations of motion become simpler

$$\begin{aligned} u_j'' + \Omega^2 u_j = 0 \\ v_j'' + \Omega^2 v_j = 0 \end{aligned} \tag{3.20}$$

The particular case turns out to be the harmonic oscillator.

We get that $A_1 = 0 \Rightarrow u_j v_j' - v_j u_j' = 0 \Rightarrow$the condition on constants U_j, V_j, ξ_j, ζ_j :

$$\sum_{j=1}^{N} \sqrt{U_j V_j} sin(\xi_j - \zeta_j) = 0 \tag{3.21}$$

∘ The case $\boldsymbol{\xi_j = \zeta_j}$ satisfies the condition. Then the final result for the fields looks

$$\begin{cases} u_j(x) = \frac{\sqrt{U_j}}{\Omega}\, sin(\Omega x + \zeta_j) \\ v_j(x) = \frac{\sqrt{V_j}}{\Omega}\, sin(\Omega x + \zeta_j) \end{cases} \tag{3.22}$$

The solution of the model looks

$$n_j(t, x) = \frac{\sqrt{2E_j}}{\Omega} e^{iA_0 t + i\gamma_j} sin(x\Omega + \zeta_j),\ tg\,\gamma_j = \sqrt{\frac{V_j}{U_j}} \tag{3.23}$$

We get uniform rotation in an N-dimensional complex space. All fields rotate with the same frequency. We can rewrite

$$\begin{aligned} &\rho_j = \frac{\sqrt{2E_j}}{\Omega} sin(x\Omega + \zeta_j) \\ &n_j = \rho_j e^{i\alpha + i\gamma_j} \\ &\alpha = A_0 t \\ &A_\mu = \partial_\mu \alpha = \begin{pmatrix} A_0 \\ 0 \end{pmatrix} \end{aligned} \tag{3.24}$$

The solution corresponds to table case 7.

∘ The case $\boldsymbol{\xi_j \neq \zeta_j}$. Then the solution looks

$$n_j(t,x) = \frac{e^{iA_0 t + i\gamma_j}}{\Omega} \left(\sqrt{U_j} sin(x\Omega + \xi_j) + i\sqrt{V_j} sin(x\Omega + \zeta_j) \right) \tag{3.25}$$

with restriction (3.21). We can rewrite

$$\begin{aligned} &\tilde{n}_j = \frac{\sqrt{U_j}}{\Omega} sin(x\Omega + \xi_j) + i\frac{\sqrt{V_j}}{\Omega} sin(x\Omega + \zeta_j) \\ &n_j = \tilde{n}_j e^{i\alpha + i\gamma_j} \\ &\alpha = A_0 t \\ &A_\mu = \partial_\mu \alpha = \begin{pmatrix} A_0 \\ 0 \end{pmatrix} \end{aligned} \tag{3.26}$$

The solution corresponds to table case 8.

Initial conditions

∘ The case $\boldsymbol{\xi_j = \zeta_j}$

Remind constraint $\bar{n}_j n_j = r \;\Rightarrow$

$$\sum_{j=1}^{N} 2E_j sin^2\left(x\Omega + \zeta_j\right) = r\Omega^2 = E_0|_{A_1=0} \tag{3.27}$$

As well as $\Omega^2 = \frac{E_0}{r}$ from (3.27) $\Rightarrow$

$$\sum_{j=1}^{N} E_j sin^2\left(x\Omega + \zeta_j\right) = \frac{E_0}{2} = \frac{1}{2}\sum_{j=1}^{N} E_j$$

$$\sum_{j=1}^{N} E_j cos\left(2x\Omega + 2\zeta_j\right) = 0$$

The expression can represent

$$cos\, 2\Omega x \left(\sum_{j=1}^{N} E_j cos\, 2\zeta_j \right) - sin\, 2\Omega x \left(\sum_{j=1}^{N} E_j sin\, 2\zeta_j \right) = 0 \tag{3.28}$$

As well as x is not constant (3.28) is right only when

$$\begin{cases} \sum_{j=1}^{N} E_j cos\, 2\zeta_j = 0 \\ \sum_{j=1}^{N} E_j sin\, 2\zeta_j = 0 \end{cases} \Leftrightarrow \sum_{j=1}^{N} E_j e^{2i\zeta_j} = 0 \tag{3.29}$$

(3.29) describes a linearly dependent system of N vectors on the complex plane

∘ The case $\boldsymbol{\xi_j \neq \zeta_j}$.

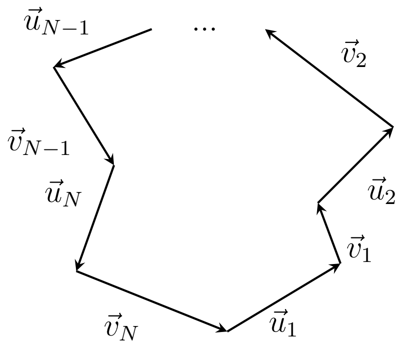
$\bar{n}_j n_j = r \overset{(3.25)}{\Rightarrow}$

$$\sum_{j=1}^{N} U_j sin^2(x\Omega + \xi_j) + V_j sin^2(x\Omega + \zeta_j) = r\Omega^2 = E_0 \tag{3.30}$$

$$cos(2\Omega x)\left(\sum_{j=1}^{N} U_j cos(2\xi_j) + V_j cos(2\zeta_j)\right) + sin(2\Omega x)\left(\sum_{j=1}^{N} U_j sin(2\xi_j) + V_j sin(2\zeta_j)\right) = 0 \tag{3.31}$$

$$\begin{cases} \sum_{j=1}^{N} U_j cos(2\xi_j) + V_j cos(2\zeta_j) = 0 \\ \sum_{j=1}^{N} U_j sin(2\xi_j) + V_j sin(2\zeta_j) = 0 \end{cases} \Rightarrow \sum_{j=1}^{N} \left(U_j e^{2i\xi_j} + V_j e^{2i\zeta_j}\right) = 0 \tag{3.32}$$

Figure 1: The linearly dependent system of vectors of initial conditions



(3.32) describes a linearly dependent system of $2N$ vectors on the complex plane (2-dimensional space). Phase displacement leads to the system of $2N$ vectors.

The Hamiltonian density

∘ $\xi_j = \zeta_j$ If we substitute (3.23) in (3.2) and take into account (3.29), we get

$$\mathcal{H} = E_0(A_0)$$

∘ $\xi_j \neq \zeta_j$ If we substitute (3.25) in (3.2) and take into account (3.32), we get

$$\mathcal{H} = E_0(A_0)$$

Therefore, constants E_j are like energies. Their sum forms the density of Hamiltonian.

$U_j \neq const$

There is a mixing of fields between the imaginary and real parts. We will look for the solution in polar coordinates.

$$\begin{aligned} u_j(x) &= \rho_j(x)\, cos(\varphi_j(x)) \\ v_j(x) &= \rho_j(x)\, sin(\varphi_j(x)) \end{aligned} \tag{3.33}$$

If we use (3.33) in (3.7) and (3.8), one can show that the system is equivalent

$$\begin{cases} \rho_j'' - \rho_j(\varphi_j')^2 + 2A_1\rho_j\varphi_j' + (\Lambda - A_0^2)\rho_j = 0 \\ 2\rho_j'\varphi_j' + \rho_j\varphi_j'' - 2A_1\rho_j' = 0 \end{cases} \tag{3.34}$$

In the form (3.33)

$$A_1 = \frac{1}{r}\sum_{j=1}^{N} \rho_j^2\varphi_j' \tag{3.35}$$

A_1 looks like the sum of sectorial speeds. (3.15) rewrites

$$(\rho_j')^2 + \rho_j^2\left((\varphi_j')^2 + \Lambda - A_0^2\right) = 2E_j > 0 \tag{3.36}$$

Notice, that (3.36) is similar to the equation of the circle $x^2 + y^2 = r^2$. We can write

$$\sqrt{2E_j} cos\,\nu_j(x) = \rho_j' \tag{3.37}$$

$$\sqrt{2E_j} sin\,\nu_j(x) = \rho_j\sqrt{(\varphi_j')^2 + \Lambda - A_0^2} \tag{3.38}$$

$\nu_j(x)$ - is unknown function. From (3.37) we find ρ_j

$$\rho_j(x) = \sqrt{2E_j}\int dx\, cos\,\nu_j(x) + \lambda_j \tag{3.39}$$

Example of "uniform rotation"

Put $\varphi_j' = const = \upsilon_j$. Then $\varphi_j = x\upsilon_j + \lambda_j$ and

$$\rho_j^2(\upsilon_j^2 + \Lambda - A_0^2) = 2E_j sin^2\nu_j(x) \;\Rightarrow$$

$$\rho_j = \sqrt{\frac{2E_j}{\upsilon_j^2 + \Lambda - A_0^2}} sin\,\nu_j(x) = \sqrt{2E_j}\int dx\, cos\,\nu_j(x) + \lambda_j$$

Differentiate by x and get

$$\frac{\nu_j' \cos\nu_j}{\sqrt{\upsilon_j^2+\Lambda+A_0^2}} = \cos\nu_j(x) \ \Rightarrow \nu_j(x) = x\sqrt{\upsilon_j^2+\Lambda-A_0^2}+\zeta_j$$

So, we find the unknown function $\nu_j(x)$ explicitly. It helps us to find ρ_j

$$\rho_j(x) = \sqrt{\frac{2E_j}{\upsilon_j^2+\Lambda-A_0^2}} sin\left(x\sqrt{\upsilon_j^2+\Lambda-A_0^2}+\zeta_j\right) \tag{3.40}$$

The exact solution looks

$$n_j(t,x) = \sqrt{\frac{2E_j}{\upsilon_j^2+\Lambda-A_0^2}} sin\left(x\sqrt{\upsilon_j^2+\Lambda-A_0^2}+\zeta_j\right) e^{iA_0t+ix\upsilon_j+i\lambda_j} \tag{3.41}$$

We should prove that it is the solution of equations. If we substitute (3.41) in (3.34) we get

$$\begin{cases} (A_1\upsilon_j-\upsilon_j^2)\ sin\left(x\sqrt{\upsilon_j^2+\Lambda-A_0^2}+\zeta_j\right)=0 \\ (\upsilon_j-A_1)\rho_j'=0 \end{cases}$$

Our function (3.41) is the solution then and only then $\upsilon_j = A_1$. So, we get that all complex fields should rotate at the same speed. The result is similar to (3.23). Fields can represent

$$n_j(t,x) = \frac{\sqrt{2E_j}}{\Omega} e^{iA_0t+iA_1x+i\lambda_j} sin\left(x\Omega+\zeta_j\right) \tag{3.42}$$

Here notation is $\Omega^2(A_0,A_1) = \Lambda - A_\mu A_\mu = \frac{E_0}{r}$, where E_0 from (3.17). We see the energy of the whole system defines the frequency of "spatial oscillations".

The real and complex parts of the function (3.42) have the same look as (3.23), so initial conditions describe by N linearly dependent vectors as well as (3.27) -(3.32).

We can rewrite

$$\begin{aligned} &\rho_j = \frac{\sqrt{2E_j}}{\Omega} sin(x\Omega+\zeta_j) \\ &n_j = \rho_j e^{i\alpha(t,x)+i\lambda_j} \\ &\alpha = A_0t+A_1x \\ &A_\mu = \partial_\mu\alpha = \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} \end{aligned} \tag{3.43}$$

The solution corresponds to table case 7. Notice that if $A_1 = 0$, then fields (3.42) become (3.23) with the same Ω. It means that the found earlier case corresponds to the uniform rotation with $A_1 = 0$.

If we substitute (3.42) in (3.2) and take into account ($\Omega^2 = \frac{E_0}{r}$, (3.29), (3.35)), we get

$$\mathcal{H} = E_0(A_0, A_1) \tag{3.44}$$

Therefore, constants E_j are like energies. Their sum forms the density of Hamiltonian.

Comments:

1) (3.44)means that the amplitudes of fields proportionate the square root of energy density.

2) Let's look at the formula (3.42), remind that $E_0 = r\Omega^2$, and write

$$n_j(t,x) = \sqrt{\frac{2}{r}}\sqrt{\frac{E_j}{E_0}}e^{i\alpha(t,x)+i\lambda_j}sin\,(x\Omega + \zeta_j) \tag{3.45}$$

The look (3.45) points out that the amplitude of a field proportional to the square root of the ratio of corresponding energy to the energy of the whole system.

Analysis of the nonlinear system

Consider (3.34) and notice that the bottom equation (3.34) can be integrated with new constants L_j

$$(\varphi_j' - A_1)\rho_j^2 = L_j = const \tag{3.46}$$

$$\varphi_j' = A_1 + \frac{L_j}{\rho_j^2} \tag{3.47}$$

The curious observation connecting with constants L_j is

$$\sum_{j=1}^{N} L_j = \sum_{j=1}^{N} \rho_j^2\varphi_j' - A_1 \sum_{j=1}^{N} \rho_j^2 = A_1 r - A_1 r = 0$$

$$\sum_{j=1}^{N} L_j = 0 \tag{3.48}$$

Constants L_j resemble angular momentum, and (3.48) resemble the law of conservation of momentum. Substitute (3.47) in (3.34) and get second-order nonlinear Ermakov equation

$$\rho_j'' + \Omega^2\rho_j = \frac{L_j^2}{(\rho_j)^3} \tag{3.49}$$

Here $\Omega^2 = \frac{E_0}{r}$, as in the example above.

It is convenient to represent our results of time-harmonic oscillations in the Table 4

Table 4: Comparing two cases of time-harmonic oscillations

Spacial "uniform" rotation	General case
$\varphi_j' = A_1$	$\varphi_j' = A_1 + \frac{L_j}{\rho_j^2}$
$\rho_j'' + \Omega^2 \rho_j = 0$	$\rho_j'' + \Omega^2 \rho_j = \frac{L_j^2}{(\rho_j)^3}$

It means that spacial uniform rotation is the limiting solution of the Ermakov equation for $L_j \to 0$.

Hence, we will demand that solutions of the Ermakov equation give us uniform rotation and fields (3.42) when $L_j = 0$.

However, we will not investigate the Ermakov equation because there is a more straightforward solution to the initial system of equations. The point is that we did a nonlinear change of variables in the linear differential system of equation. The construction above helps us to make a meaningful interpretation of the case and describe the constants of integration with unique properties.

3.1.2 The solution of the case

Equations (3.7) and (3.8) can trivially solve through the replacement

$$\begin{aligned} u_j' &= w \\ v_j' &= s \end{aligned} \tag{3.50}$$

Then we rewrite

$$\frac{d}{dx}\begin{pmatrix} u_j \\ v_j \\ w_j \\ s_j \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ A_0^2 - \Lambda & 0 & 0 & -2A_1 \\ 0 & A_0^2 - \Lambda & 2A_1 & 0 \end{pmatrix} \begin{pmatrix} u_j \\ v_j \\ w_j \\ s_j \end{pmatrix} \tag{3.51}$$

The eigenvalues λ of the matrix above are $\lambda_{1,2,3,4} = \pm iA_1 \pm i\Omega$ ($\Omega^2 = A_1^2 - A_0^2 + \Lambda$). So the solution

$$\begin{pmatrix} u_j \\ v_j \\ w_j \\ s_j \end{pmatrix}(x) = e^{i\Omega x + iA_1 x}\begin{pmatrix} u_j^1 \\ v_j^1 \\ w_j^1 \\ s_j^1 \end{pmatrix} + e^{i\Omega x - iA_1 x}\begin{pmatrix} u_j^2 \\ v_j^2 \\ w_j^2 \\ s_j^2 \end{pmatrix} + e^{-i\Omega x + iA_1 x}\begin{pmatrix} u_j^3 \\ v_j^3 \\ w_j^3 \\ s_j^3 \end{pmatrix} + e^{-i\Omega x - iA_1 x}\begin{pmatrix} u_j^4 \\ v_j^4 \\ w_j^4 \\ s_j^4 \end{pmatrix} \tag{3.52}$$

where $u_j^i, v_j^i, w_j^i, s_j^i \in \mathbb{C} - const,\ i \in \overline{1,4}$. From (3.50) follows how constants u_i, v_i and w_i, s_i are connected. Also we know, that functions $u_j(x)$ and $v_j(x)$ are real, so

$$\begin{cases} u_j(x) = \bar{u}_j(x) \\ v_j(x) = \bar{v}_j(x) \end{cases} \Rightarrow \begin{cases} u_j^1 = a_j^1 e^{i\varphi_j^1} = \bar{u}_j^4, \quad v_j^1 = b_j^1 e^{i\psi_j^1} = \bar{v}_j^4 \\ u_j^2 = a_j^2 e^{i\varphi_j^2} = \bar{u}_j^3, \quad v_j^2 = b_j^2 e^{i\psi_j^2} = \bar{v}_j^3 \end{cases}$$

From that we find

$$n_j(t,x) = e^{i\omega t}(a_j^1\, cos\left(\Omega x + A_1 x + \varphi_j^1\right) + ib_j^1\, cos\left(\Omega x + A_1 x + \psi_j^1\right) + \\ + a_j^2\, cos\left(\Omega x - A_1 x + \varphi_j^2\right) + ib_j^2\, cos\left(\Omega x - A_1 x + \psi_j^2\right)) \tag{3.53}$$

Equivalent form is

$$n_j(t,x) = e^{iA_0 t}\left(z_1 cos\left(\Omega x + A_1 x\right) + z_2 sin\left(\Omega x + A_1 x\right) + z_3 cos\left(\Omega x - A_1 x\right) + z_4 sin\left(\Omega x - A_1 x\right)\right) \tag{3.54}$$

$z_1, z_2, z_3, z_4 \in \mathbb{C}$

3.2 $CP(1)$ and amplitude rotation

Shifman and Yung, in their study, pointed out to the correspondence between BPS kinks in $CP(1)$ and confined monopoles. That link motivates the search of exact solutions for the $CP(1)$ model. We focus on solutions with nonlinear amplitude and phase dependence. A class of such solutions has found up to one unknown function.

$\mid n_1 \mid^2 + \mid n_1 \mid^2 = r$ - equation of circle. So we can write

$$n_1 = \sqrt{r}\, cos\, \alpha(t,x)\, e^{i\varphi_1(t,x)} \tag{3.55}$$

$$n_2 = \sqrt{r}\, sin\, \alpha(t,x)\, e^{i\varphi_2(t,x)} \tag{3.56}$$

$$A_\mu \overset{(1.6)}{=} cos^2\alpha\, \partial_\mu\varphi_1 + sin^2\alpha\, \partial_\mu\varphi_2 \tag{3.57}$$

If we substitute (3.55), (3.56) in (1.11) we get the system

$$\begin{cases} -\left(\frac{1}{2}(\partial_\mu\varphi_1)^2 + \frac{1}{2}\partial_\mu\varphi_1\partial_\mu\varphi_2 + \Lambda - (\partial_\mu\alpha)^2\right) cos\,\alpha + \Box\alpha\, sin\,\alpha - \frac{1}{2}\left((\partial_\mu\varphi_1)^2 - \partial_\mu\varphi_1\partial_\mu\varphi_2\right) cos\,3\alpha = 0 \\ -\Box\,\varphi_1 cos\,\alpha + \frac{3}{2}\partial_\mu\alpha\partial_\mu(\varphi_1-\varphi_2) sin\,\alpha - \frac{1}{2}\left(\partial_\mu\alpha\partial_\mu(\varphi_1-\varphi_2)\right) sin\,3\alpha = 0 \\ -\Box\,\alpha\, cos\,\alpha - \left(\frac{1}{2}(\partial_\mu\varphi_2)^2 + \frac{1}{2}\partial_\mu\varphi_1\partial_\mu\varphi_2 + \Lambda - (\partial_\mu\alpha)^2\right) sin\,\alpha + \frac{1}{2}\left((\partial_\mu\varphi_2)^2 - \partial_\mu\varphi_1\partial_\mu\varphi_2\right) sin\,3\alpha = 0 \\ \frac{3}{2}\partial_\mu\alpha\partial_\mu(\varphi_1-\varphi_2) cos\,\alpha - \Box\varphi_2 sin\,\alpha + \frac{1}{2}\left(\partial_\mu\alpha\partial_\mu(\varphi_1-\varphi_2)\right) cos\,3\alpha = 0 \end{cases} \tag{3.58}$$

Zero coefficients

If we annihilate the coefficients in front of $sin\,\alpha$, $cos\,\alpha$, $sin\,3\alpha$, $cos\,3\alpha$, we get another system

$$\begin{cases} \Box\alpha = 0 \quad \partial_\mu\alpha\partial_\mu(\varphi_1-\varphi_2) = 0 \\ \Box\varphi_1 = 0 \quad \partial_\mu\varphi_1\partial_\mu(\varphi_1-\varphi_2) = 0 \\ \Box\varphi_2 = 0 \quad \partial_\mu\varphi_2\partial_\mu(\varphi_1-\varphi_2) = 0 \\ \frac{1}{2}(\partial_\mu\varphi_2)^2 + \frac{1}{2}\partial_\mu\varphi_1\partial_\mu\varphi_2 + \Lambda - (\partial_\mu\alpha)^2 = 0 \\ \frac{1}{2}(\partial_\mu\varphi_1)^2 + \frac{1}{2}\partial_\mu\varphi_1\partial_\mu\varphi_2 + \Lambda - (\partial_\mu\alpha)^2 = 0 \end{cases} \tag{3.59}$$

From that, we can get

I.

$$(\partial_\mu\varphi_1)^2 = \partial_\mu\varphi_1\partial_\mu\varphi_2 = (\partial_\mu\varphi_2)^2 = (\partial_\mu\alpha)^2 - \Lambda \tag{3.60}$$

II. For the substitution (3.55) and (3.56) Lorentz calibration is fulfilled automatically

$$\partial_\mu A_\mu = 0$$

III.

$$A_\mu A_\mu = (\partial_\mu \alpha)^2 - \Lambda \tag{3.61}$$

IV. It can directly prove from (3.60)

$$\varphi_1 = \varphi + c_1$$
$$\varphi_2 = \varphi + c_2, \; c_1, c_2 \in \mathbb{R}, \Box\varphi = 0$$

So,

$$n_1 = \sqrt{r}\, cos\, \alpha\, e^{i\varphi + ic_1} \tag{3.62}$$
$$n_2 = \sqrt{r}\, sin\, \alpha\, e^{i\varphi + ic_2} \tag{3.63}$$

where α and φ connect through the relation

$$\partial_\mu \varphi \partial_\mu \varphi \overset{(3.60)}{=} \partial_\mu \alpha \partial_\mu \alpha - \Lambda \overset{(3.61)}{=} A_\mu A_\mu \tag{3.64}$$

It shows that

$$A_\mu = \partial_\mu \varphi \tag{3.65}$$

$$\begin{cases} \Box\alpha = 0 \\ \Box\varphi = 0 \end{cases} \Rightarrow \begin{cases} \alpha = \alpha_1(t-x) + \alpha_2(t+x) \\ \varphi = \beta_1(t-x) + \beta_2(t+x) \end{cases}$$

(3.64) is equivalent

$$4\alpha_1'(t-x)\alpha_2'(t+x) - \Lambda = 4\beta_1'(t-x)\beta_2'(t+x) \tag{3.66}$$

Take into account, that if we have only one wave α_1 or α_2 then $\partial_\mu \alpha \partial_\mu \alpha = 0$. Similarly, if we have only β_1 or β_2 then $\partial_\mu \varphi \partial_\mu \varphi = 0$

(3.62) and (3.63) can be rewritten

$$n_1 = \sqrt{r}\, cos(\alpha_1(t-x) + \alpha_2(t+x))\, e^{i(\beta_1(t-x)+\beta_2(t+x)+c_1)} \tag{3.67}$$
$$n_2 = \sqrt{r}\, sin(\alpha_1(t-x) + \alpha_2(t+x))\, e^{i(\beta_1(t-x)+\beta_2(t+x)+c_2)} \tag{3.68}$$

Let's pay attention to (3.66). If we go to other variables

$$u = t - x$$
$$v = t + x$$

we can get, that

$$\alpha_1''(u)\alpha_2'(v) = \beta_1''(u)\beta_2'(v) \tag{3.69}$$
$$\alpha_1'(u)\alpha_2''(v) = \beta_1'(u)\beta_2''(v) \tag{3.70}$$

If we use that relations and (3.66), we can show, that either $\alpha_1'(u) = const$ or $\alpha_2'(v) = const$. If $\alpha_2'(v) = const$ we can find $(C_1, C_2 \in \mathbb{R})$

$$\begin{array}{ll} \alpha_1'(u) = C_1 + C_2 b(u) & \alpha_1 = C_1(t-x) + C_2 \int_0^{t-x} b(u)du \\ \alpha_2'(v) = \frac{\Lambda}{4C_1} & \alpha_2 = \frac{\Lambda}{4C_1}(t+x) \\ \beta_1'(u) = b(u) & \beta_1 = \int_0^{t-x} b(u)du \\ \beta_2'(v) = \frac{\Lambda C_2}{4C_1} & \beta_2 = \frac{\Lambda C_2}{4C_1}(t+x) \end{array}$$

And the fields are equal

$$n_1 = \sqrt{r}\, cos\left(C_1(t-x) + C_2 \int\limits_0^{t-x} b(u)du + \frac{\Lambda}{4C_1}(t+x) \right) e^{i\left(\int_0^{t-x} b(u)du + \frac{\Lambda C_2}{4C_1}(t+x) + c_1\right)}$$
$$n_2 = \sqrt{r}\, sin\left(C_1(t-x) + C_2 \int\limits_0^{t-x} b(u)du + \frac{\Lambda}{4C_1}(t+x) \right) e^{i\left(\int_0^{t-x} b(u)du + \frac{\Lambda C_2}{4C_1}(t+x) + c_2\right)}$$

$b(u)$ is unknown function

$$A_\mu = \partial_\mu \varphi \Rightarrow A_\mu = \begin{pmatrix} \varphi_t \\ \varphi_x \end{pmatrix} = \begin{pmatrix} \beta_1'(t-x) + \beta_2'(t+x) \\ -\beta_1'(t-x) + \beta_2'(t+x) \end{pmatrix}$$
$$A_\mu = \begin{pmatrix} \frac{\Lambda C_2}{4C_1} + b(t-x) \\ \frac{\Lambda C_2}{4C_1} - b(t-x) \end{pmatrix} \tag{3.71}$$

3.3 $CP(1)$ and Hopf equation

3.3.1 Connection of the gauge field with Hopf equation

Let's add the restriction on the Lorentz gauge in 1+1 dimensions

$$\begin{cases} \partial_\mu A_\mu = 0 \\ A_\mu A_\mu = -\frac{\Lambda}{2} \end{cases} \tag{3.72}$$

That restriction is analog (2.3). In m spacial dimensions, there are particular solutions below Table 3 of central cases. But we can't do the suggested substitution in one spatial dimension, because in that case, there is only one phase function ψ for both fields.

$$\begin{aligned} n_1 &= r_1 e^{i\psi} \\ n_2 &= r_2 e^{-i\psi} \end{aligned} \tag{3.73}$$

If we make suggested substitution $\psi = \psi(\omega_\mu x_\mu)$, $\omega_\mu\omega_\mu = 0$, then $A_\mu \sim \omega_\mu$, and equation of motion $\partial_\mu \Psi \partial_\mu \Psi - 2A_\mu \partial_\mu \Psi = \Lambda E$ is reduced to the relation $\Lambda = 0$, what can't be. So, we can't use the substitution $\psi = \psi(\omega_\mu x_\mu)$.

In 1+1 dimensions, restrictions (3.72) lead to the interesting connection with the quasi-linear Hopf equation. Let's consider the gauge field A_μ

$$\vec{A} = \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

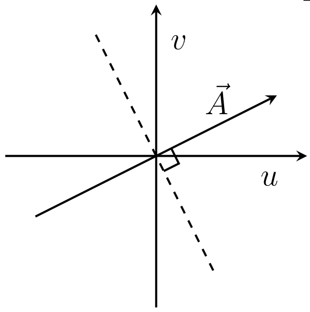
$$\partial_\mu A_\mu = 0 \Leftrightarrow u'_t = v'_x \tag{3.74}$$

$$A_\mu A_\mu = -\frac{\Lambda}{2} \Leftrightarrow u^2 - v^2 = -\frac{\Lambda}{2} \tag{3.75}$$

$$(3.75) \Rightarrow \begin{cases} u'_t u = v'_t v \\ u'_x u = v'_x v \end{cases} \overset{(3.74)}{\Rightarrow} \begin{cases} v'_x u = v'_t v \\ u'_x u = u'_t v \end{cases} \Rightarrow \begin{cases} \vec{A} = \begin{pmatrix} u \\ v \end{pmatrix} \perp \begin{pmatrix} v'_x \\ -v'_t \end{pmatrix} \\ \vec{A} = \begin{pmatrix} u \\ v \end{pmatrix} \perp \begin{pmatrix} u'_x \\ -u'_t \end{pmatrix} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{pmatrix} v'_x \\ -v'_t \end{pmatrix} || \begin{pmatrix} u'_x \\ -u'_t \end{pmatrix} \Rightarrow C(t,x) \begin{pmatrix} u'_x \\ -u'_t \end{pmatrix} = \begin{pmatrix} v'_x \\ -v'_t \end{pmatrix} \overset{(3.74)}{=} \begin{pmatrix} u'_t \\ -v'_t \end{pmatrix} \Rightarrow \begin{cases} u'_t = C u'_x \\ v'_x = C u'_x \\ v'_t = C u'_t = C^2 u'_x \end{cases} \Rightarrow$$

Figure 2: Vector $\vec{A}$ on the plane



$$\Rightarrow \begin{cases} \begin{pmatrix} v'_x \\ -v'_t \end{pmatrix} = \begin{pmatrix} C \\ -C^2 \end{pmatrix} u'_x \\ \begin{pmatrix} u \\ v \end{pmatrix} \perp \begin{pmatrix} v'_x \\ -v'_t \end{pmatrix} \end{cases} \Rightarrow \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} C(t,x) \\ 1 \end{pmatrix} f(t,x) \Rightarrow \\ \Rightarrow v = f \Rightarrow u = Cf = C\nu$$

$$u^2 - v^2 = (C^2-1)v^2 = -\frac{\Lambda}{2} = const \neq 0 \Rightarrow \begin{cases} v(t,x) \neq 0 \\ C(t,x) \neq \pm 1 \end{cases}$$

If we suppose $C^2 < 1,\ \Lambda > 0$ (alternative variant is $C^2 > 1,\ \Lambda < 0$), the fields u and v look

$$u = \frac{C}{\sqrt{1-C^2}}\sqrt{\frac{\Lambda}{2}} \tag{3.76}$$

$$v = \frac{1}{\sqrt{1-C^2}}\sqrt{\frac{\Lambda}{2}} \tag{3.77}$$

And $C = C(t,x)$ satisfies the quasi-linear equation of the Hopf

$$C'_t = CC'_x \tag{3.78}$$

So, we get that the gauge field with additional restriction depends on the function, satisfying Hopf equation, and has a view

$$\vec{A} = \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} = \frac{1}{\sqrt{1-C^2}}\sqrt{\frac{\Lambda}{2}} \begin{pmatrix} C(t,x) \\ 1 \end{pmatrix} \tag{3.79}$$

Notice the relationship between time and spatial gauge fields

$$A_0 = CA_1 \tag{3.80}$$

3.3.2 Table case 2 for $CP(1)$ with additional restriction

Consider two complex fields (3.73), where $\psi = \psi(t, x)$, with restriction (3.72). Then (if $r_1 \neq r_2$)

$$A_\mu = \frac{r_1^2 - r_2^2}{r} \partial_\mu \psi = \frac{r_1^2 - r_2^2}{r} \begin{pmatrix} \psi'_t \\ \psi'_x \end{pmatrix} \overset{(3.79)}{=} \frac{1}{\sqrt{1 - C^2(t,x)}} \sqrt{\frac{\Lambda}{2}} \begin{pmatrix} C(t,x) \\ 1 \end{pmatrix} \tag{3.81}$$

We investigate the case $C^2(t, x) < 1,\ \Lambda > 0$. From (3.80) we have

$$\psi'_t = C(t, x)\psi'_x \tag{3.82}$$

Where $C(t, x)$ satisfies the Hopf equation $C'_t = CC'_x$. So, we can suggest that

$$\psi = C(t, x) \tag{3.83}$$

and integrate differential equations given from (3.81)

$$\frac{r_1^2 - r_2^2}{r} \begin{pmatrix} C'_t \\ C'_x \end{pmatrix} \overset{(3.79)}{=} \frac{1}{\sqrt{1 - C^2(t,x)}} \sqrt{\frac{\Lambda}{2}} \begin{pmatrix} C(t,x) \\ 1 \end{pmatrix}$$

Define $\lambda = \frac{r}{r_1^2 - r_2^2}\sqrt{\frac{\Lambda}{2}}$

$$\begin{cases} \frac{C'_t}{C}\sqrt{1 - C^2(t,x)} = \lambda \\ C'_x\sqrt{1 - C^2(t,x)} = \lambda \end{cases} \tag{3.84}$$

Remind $C^2 < 1,\ \Lambda > 0$,

$$\begin{cases} \sqrt{1 - C^2} + ln\left(\frac{C}{1+\sqrt{1-C^2}}\right) = \lambda t + k_1(x) \\ C\sqrt{1 - C^2} + arcsin\left(\frac{C}{1+\sqrt{1-C^2}}\right) = 2\lambda x + k_2(t) \end{cases} \tag{3.85}$$

$k_1(x)$ and $k_2(x)$ are some functions, which we get after integration. Substitute $C(t, x) = th\,\gamma(t, x)$ in equations of motion

$$\begin{cases} \frac{1}{ch\,\gamma(t,x)} + ln\,th\frac{\gamma(t,x)}{2} = \lambda t + k_1(x) \\ \frac{th\,\gamma(t,x)}{ch\,\gamma(t,x)} + arcsin\,th\frac{\gamma(t,x)}{2} = 2\lambda x + k_2(t) \end{cases} \tag{3.86}$$

and in the Hopf equation (3.78)

$$\gamma'_t = \gamma'_x th\,\gamma \tag{3.87}$$

If we take time and space partial derivatives from (3.86) and use (3.87), it can show that $th\,\gamma$ should be equal a constant, what leads to the contradiction. However, the situation changes if we suppose that the Λ is a function such $\Lambda = \Lambda(t, x)\,|\,\forall t, x \hookrightarrow \Lambda(t, x) \neq 0$. Further investigation is necessary.

3.3.3 Table case 4 for $CP(1)$ with additional restriction

$$\begin{pmatrix} A_0 \\ A_1 \end{pmatrix} = \frac{1}{\sqrt{1-C^2(t,x)}}\sqrt{\frac{\Lambda}{2}}\begin{pmatrix} C(t,x) \\ 1 \end{pmatrix} = \begin{pmatrix} (\rho_1^2-\rho_2^2)\varphi_t' \\ (\rho_1^2-\rho_2^2)\varphi_x' \end{pmatrix}$$

From (3.80)

$$\varphi_t' = C\varphi_x' \tag{3.88}$$

If we choose $\varphi = C$,then (3.88) is fulfilled and

$$n_1 = \rho_1 e^{i\alpha+iC} \tag{3.89}$$

$$n_2 = \rho_2 e^{i\alpha-iC} \tag{3.90}$$

The system of equations of motion has the view

$$\begin{cases} (\partial_\mu\alpha\partial_\mu\alpha-\Lambda)\,cos\,\alpha + \Box\alpha\,sin\,\alpha - \partial_\mu C\partial_\mu C\,cos\,3\alpha = 0 \\ -\Box\,C\,cos\,\alpha + 3\partial_\mu\alpha\partial_\mu C\,sin\,\alpha - \partial_\mu\alpha\partial_\mu C\,sin\,3\alpha = 0 \\ -\Box\,\alpha\,cos\,\alpha + (\partial_\mu\alpha\partial_\mu\alpha-\Lambda)\,sin\,\alpha - \partial_\mu C\partial_\mu C\,sin\,3\alpha = 0 \\ 3\partial_\mu\alpha\partial_\mu C\,cos\,\alpha + \Box C\,sin\,\alpha + \partial_\mu\alpha\partial_\mu C\,cos\,3\alpha = 0 \end{cases} \tag{3.91}$$

Define $\varsigma = \partial_\mu\alpha\partial_\mu\alpha - \Lambda$ and rewrite the system in the matrix form

$$\begin{pmatrix} \varsigma & \Box\alpha & -\partial_\mu C\partial_\mu C & 0 \\ \Box\alpha & -\varsigma & 0 & \partial_\mu C\partial_\mu C \\ 3\partial_\mu\alpha\partial_\mu C & \Box C & \partial_\mu\alpha\partial_\mu C & 0 \\ \Box C & -3\partial_\mu\alpha\partial_\mu C & 0 & \partial_\mu\alpha\partial_\mu C \end{pmatrix}\begin{pmatrix} cos\,\alpha \\ sin\,\alpha \\ cos\,3\alpha \\ sin\,3\alpha \end{pmatrix} = 0 \tag{3.92}$$

The system has a block structure. Such matrix form with nonlinear vectors is remarkable.

3.3.4 Possible nontrivial topological charge of gauge Hopf field

In spite of it hasn't been picked up relevant functions n_1 and n_2, satisfying equations of motion and giving certain A_μ ($A_\mu A_\mu = \varLambda$), we can explore corresponding topological current

$$F_{\mu\nu} = \begin{pmatrix} 0 & F_{01} \\ -F_{01} & 0 \end{pmatrix}$$

The case $C^2 < 1,\ \Lambda > 0$

$$\begin{pmatrix} A_0 \\ A_1 \end{pmatrix} = \frac{1}{\sqrt{1-C^2(t,x)}}\sqrt{\frac{\Lambda}{2}}\begin{pmatrix} C(t,x) \\ 1 \end{pmatrix} \tag{3.93}$$

$$F_{01} = -\frac{C'_x}{\sqrt{1 - C^2(t,x)}}\sqrt{\frac{\Lambda}{2}} \tag{3.94}$$

$$Q = \int\limits_{\mathbb{R}^2} dt\, dx\, \epsilon_{\mu\nu} F_{\mu\nu} = -\sqrt{2\mathit{\Lambda}}\int\limits_{\mathbb{R}^2} dt\, dx \frac{C'_x}{\sqrt{1 - C^2(t,x)}} =$$

$$Q = -\sqrt{2\mathit{\Lambda}}\int\limits_{\mathbb{R}} dt\, \left(arcsin\, C(t,\infty) - arcsin\, C(t,-\infty)\right) \tag{3.95}$$

The case $C^2 > 1,\ \Lambda < 0$. We change $\Lambda \to -\Lambda$ and rewrite

$$\vec{A} = \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} = \frac{1}{\sqrt{C^2(t,x) - 1}}\sqrt{\frac{\Lambda}{2}} \begin{pmatrix} C(t,x) \\ 1 \end{pmatrix} \tag{3.96}$$

where $C'_t = CC'_x$.

$$F_{01} = -\frac{C'_x}{\sqrt{C^2(t,x) - 1}}\sqrt{\frac{\Lambda}{2}} \tag{3.97}$$

$$Q = \int\limits_{\mathbb{R}^2} dt\, dx\, \epsilon_{\mu\nu} F_{\mu\nu} = -\sqrt{2\mathit{\Lambda}}\int\limits_{\mathbb{R}^2} dt\, dx \frac{C'_x}{\sqrt{C^2(t,x) - 1}} =$$

$$Q = -\sqrt{2\mathit{\Lambda}}\int\limits_{\mathbb{R}} dt\, ln\left[\frac{C(t,\infty) + \sqrt{C^2(t,\infty) - 1}}{C(t,-\infty) + \sqrt{C^2(t,-\infty) - 1}}\right] \tag{3.98}$$

(3.95) and (3.98) can not be zero in depends on the choice $C(t,x)$. These reasons point out to topologically nontrivial set solutions in 1+1 dimensions.

Conclusion

In this work, we investigate nonlinear $CP(N-1)$ and its particular solutions in classic field theory. We described the model through the matrix form, defined the central cases, and investigated which solutions can correspond in every situation. The obtained results represent extreme cases.

There is a more detailed research of the $1+1$ dimension, in which we focused on three specific cases. The first case represents the time-harmonic oscillations.

- Constants of integration corresponding Hamiltonian density obtained;
- the lack of mixing between real and imaginary parts of the fields leads to the harmonic oscillator;
- "uniform" mixing between real and imaginary parts of fields occurs with the same for all fields frequency and leads to the harmonic oscillator;
- arbitrary mixing between real and imaginary parts of fields corresponds to the nonlinear Ermakov equation of motion;
- the general solutions obtained.

The second case describes some solutions in $CP(1)$. The idea grounds on the fact that amplitudes of both fields have to be on the circle.

The third case investigates solutions in the $1+1$ dimension with a fixed vector's length of the gauge field.

- In the $1+1$ dimension, the gauge field depends on the function satisfying nonlinear Hopf equation;
- arguments about the possible existence of corresponding topologically nontrivial solutions lead.

Further investigation may focus on

- the search of exact solutions for topologically nontrivial fields configurations in the third particular case, connected with the Hopf equation;
- the research of the role appeared differential equations in the model;
- the investigation of extreme cases for higher dimensions;
- the link of obtained solutions with quantum theory.

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